Whitepaper

Recognizing Misconceptions as Opportunities

for Learning Mathematics with Understanding

Mark Ellis, PH.D., NBCT
Professor of Education at California State University, Fullerton

i-Ready Classroom Mathematics

About the Author

Mark Ellis, Ph.D., NBCT

Mark is a professor of Education at California State University, Fullerton and a highly regarded scholar and teacher of K–12 mathematics education. With more than 40 publications to his credit, Mark has served on the National Council of Teachers of Mathematics (NCTM) Board of Directors and Executive Committee and recently published a book about creating productive mathematics learning environments for elementary students. He is best known for his collaborative work with educators in developing strategies that help students understand concepts, supporting culturally relevant instruction, and addressing issues of equity in mathematics education.

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Introduction

Learning mathematics with understanding requires that students routinely engage in processes of reasoning and sense making. Rather than fixating on whether a particular answer is right or wrong, it is more productive for students to develop the habit of explaining and justifying their strategies and making revisions to their thinking when faced with information that challenges prior beliefs.

In fact, the best learning happens when students analyze their approach and revise their thinking based on new insights. This is at the heart of learning from misconceptions[1].

In this paper, we will discuss how to anticipate, elicit, and make use of student misconceptions to further students' learning of mathematics with understanding.

Let's first clarify what is meant by a misconception. As Lucariello and Naff (2014) explain:

> ". . . erroneous understandings are termed alternative conceptions or misconceptions (or intuitive theories). Alternative conceptions (misconceptions) are not unusual. In fact, they are a normal part of the learning process. We quite naturally form ideas from our everyday experience, but obviously not all the ideas we develop are correct . . ."

What distinguishes a misconception from an error or misapplication is the root of the mistake; it is a student's understanding of the concept or relationship that is incomplete or incorrect. In other words, it is a conceptual misunderstanding, not an error in calculation (see Figure 1).

Rather than fixating on whether a particular answer is right or wrong, it is more productive for students to develop the habit of explaining and justifying their strategies and making revisions to their thinking when faced with information that challenges prior beliefs.

[1]The term "misconceptions" is sometimes replaced with "alternative conceptions" or "preconceptions" as a way to avoid the negative connotation of the prefix "mis-." However, because "misconception" is still widely used in mathematics education research and curricular materials, this term will be used throughout the paper and is not meant to connote a deficit view. In fact, misconceptions are beautiful windows into students' thinking that allow for deeper learning when intentionally elicited, explored, revised, and resolved through students' sense making.

Examples of Common Errors vs. Common Misconceptions

Misconceptions in mathematics are inevitable and often predictable. It is how teachers help students recognize their misconceptions and provide opportunities to revise or replace their thinking that is crucial to supporting deep learning.

Grade 1 Example

Common Error

$$\begin{array}{r} 28 \\ +\ 47 \\ \hline 76 \end{array}$$

Explanation:
I added 8 plus 7 and got 16, so I wrote the 6 in the ones place, and then added the 1 ten to 4 plus 2 to get 7 in the tens place.

Error:
Thinking 8 + 7 is 16, not 15

Common Misconception

$$\begin{array}{r} 28 \\ +\ 47 \\ \hline 615 \end{array}$$

Explanation:
I added 8 plus 7 to get 15 and then added 2 + 4 to get 6. I wrote the 6 next to the 15 since it was in the next place value.

Misconception:
Not understanding place value

Figure 1. Examples of Common Errors vs. Common Misconceptions

Grade 4 Example

Common Error

$$\frac{3}{4} + \frac{1}{8}$$

$$\frac{5}{8} + \frac{1}{8} = \frac{6}{8}$$

Explanation:
I had to make three-fourths into eighths to have common units in the denominator, so I made each one-fourth into two-eighths. Three plus two is five, so I got five-eighths. Then I added five-eighths and one-eighths to get six-eighths.

Error:
Renaming 3/4 as 5/8 by adding 2 to the numerator rather than multiplying the numerator by 2

Common Misconception

$$\frac{3}{4} + \frac{1}{8} = \frac{4}{12}$$

Explanation:
I added the two numbers on top, 3 plus 1, and got 4. Then I added the two numbers on the bottom, 4 plus 8, and got 12.

Misconception:
Not realizing you need a common unit (denominator) to add fractions

Reasons for Misconceptions

There are many reasons students develop misconceptions. Two are described in this paper:

1. Students apply prior knowledge to new concepts incorrectly.
2. Adults say things in ways that may create misconceptions.

Some examples of each of these are discussed in the next few pages.

Applying Prior Knowledge Incorrectly

Misconceptions reflect students' efforts to apply prior knowledge, experience, and understanding to new situations.

Think about a student who knows that when multiplying 15 x 4, the commutative and distributive properties allow them to create an equivalent expression such as 15 (2 + 2), which then allows for the operation to be split into two easy-to-handle partial products, (15 x 2) and (15 x 2).

When students begin to explore and make sense of division, it is entirely natural and reasonable for them to think that 80 ÷ 4 might be expressed as 80 ÷ (2 + 2) and that this leads to the expression (80 ÷ 2) + (80 ÷ 2).

Focusing on What Students Know and Can Do

While we know from abundant research that all students think mathematically and can make sense of mathematical concepts and relationships, there is an unfortunate and destructive habit in mathematics education to refer to students through deficit language.

Terms such as "low student" and "struggling learner" do not acknowledge the wealth of knowledge, ideas, and capacity for reasoning that students possess. Assigning labels to students that convey a deficit in their ability to make sense of mathematics reflects and reinforces a fixed mindset. Further, this does harm to students' mathematical identity and distracts educators from working to find ways to support student success by building on their strengths rather than "fixating" on perceived defects.

What is more productive for students and educators alike is to adopt and communicate an asset-based perspective toward students' ability to learn mathematics. An asset-based approach to learning focuses attention and effort on identifying what students know and can do, on valuing students' strengths as learners, and on providing the strategies and resources that will support their mathematical growth.

Although incorrect, this stems from a reasonable attempt to extend one's understanding of multiplication to the operation of division. What's important is that students are encouraged to try out such ideas and are given the space to discuss and debate whether such claims are true in every case.

In trying to explain and justify a claim such as the one above, students come to realize—often with help from their peers and certainly their teacher—that the distributive property does not always hold true when working with division.

As part of this process of explaining, justifying, and revising one's thinking, it is helpful to provide students with multiple representations through which to examine and explore the concept or relationship. In the division example, students might be encouraged to use base-ten blocks, draw array models, and/or set it in a familiar context to support their sense making (see Figure 2).

The teacher might then use student-generated examples to lead a whole class discussion about why the distributive property holds true when partitioning the divisor (e.g., 80 ÷ 4 = (40 + 40) ÷ 4) but does not hold true when partitioning the dividend (because, unlike multiplication, division is not commutative).

For a video of a teacher engaging students in grappling with this misconception, see: Learn.TeachingChannel.com/Video/Common-Core-Teaching-Division

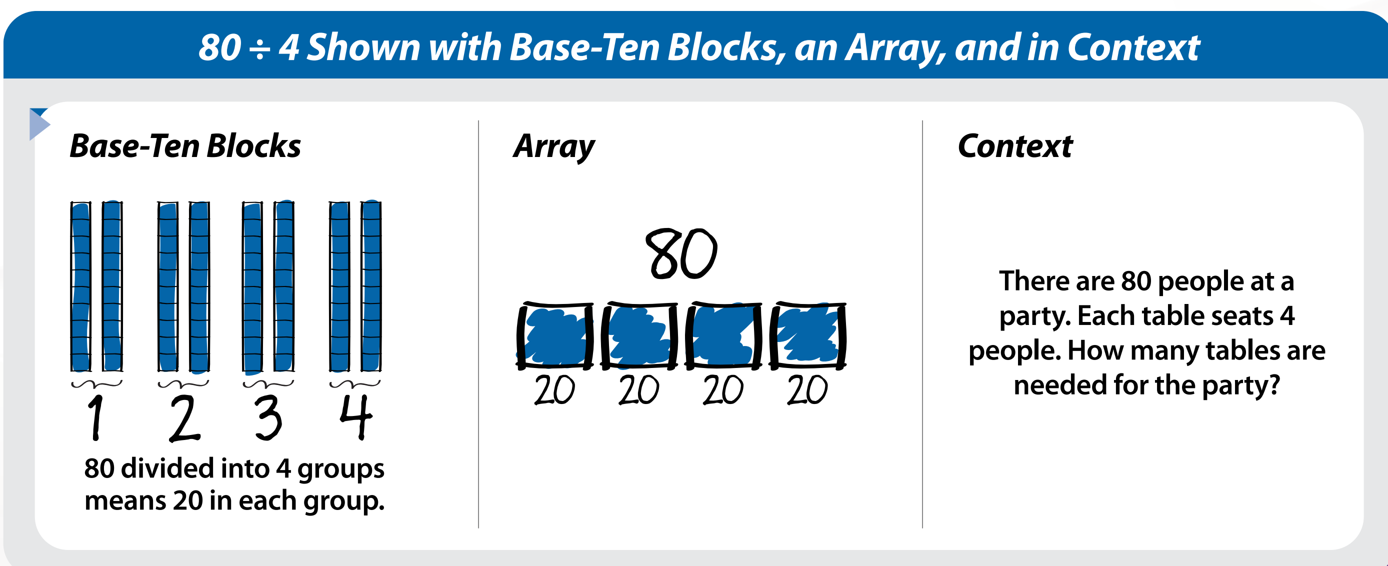


Figure 2. Student Sense Making

Things Adults Say That May Create Misconceptions for Students

Students may develop misconceptions because we as adults do not always use mathematical vocabulary accurately when we talk about familiar concepts. For example, because numbers and arithmetic operations are familiar to adults, we may be more casual in how we talk about them. In conversations between adults, this is generally not a problem.

However, when interacting with students who have yet to develop full understandings of key mathematical concepts and relationships, this lack of precision can create or reinforce misconceptions. Here are a few examples that illustrate this issue along with suggestions for how to revise the way things are phrased.

Example 1

Common Pitfall:	**Reading an equal sign as "is what" or "gives" rather than "is the same as" or "is equivalent to."** For example, 3 x 7 = ____ might be read aloud to students as, "3 times 7 gives what?" This communicates the misconception that the equal sign is an operation (something to do) rather than a relationship (something to compare).
Best Practice:	A more precise reading would be, "3 times 7 is equivalent to ____?" or "3 times 7 is the same as _____?" Each of these communicates that the equal sign represents a relationship of equivalence between the expressions on either side of it. Two additional strategies to reinforce the correct meaning of the equal sign are (a) placing the missing expression to the left rather than to the right of the equal sign (e.g., ____ = 3 x 7) and (b) posing true/false statements that students must evaluate and explain, for example, "True or False: 3 x 7 = 7 + 7 + 7."

Example 2

Common Pitfall:	**Telling students that "subtraction makes things smaller."** Certainly, when subtraction is first experienced by learners in primary grades, most of the cases they encounter in and out of school support this claim. However, one only needs to think ahead to operations with integers in upper elementary and middle school to realize the problematic nature of this statement. Just consider 3 – 0 . . . the result is not smaller.
Best Practice:	It is more precise and more helpful to say that subtraction is about taking away (counters are useful with this) and about finding the distance between two values (a number line model is particularly helpful with this); so 7 – 3 can be thought of as, "What is the result if I take 3 away from 7?" and "How far is 7 away from 3?" This will better prepare students for their later extension of understanding subtraction with integers. A similar issue with telling students that "division makes things smaller" is a good topic to ponder and discuss with other educators.

Example 3

Common Pitfall:	**Teaching students that "multiplying by 10 (or any positive power of 10) means we add zeros."** This does long-term damage to students' understanding of place value, something they must know well in order to make sense of operations with decimal numbers.
Best Practice:	Although it might appear that the product of 17 x 10 is found by "adding a zero" to get 170, what is really happening is much richer. If you can imagine a place value chart with a column for hundreds, tens, and ones, the number 17 will have the "1" in the tens column and "7" in the ones column. Multiplying 17 x 10 results in the "1" shifting from the tens column one place value to the left to become 1 hundred and the 7 moving from the ones to the tens column to become 7 tens (or 70). The 0 is placed in the ones column to represent that there are no ones. So it is more precise and more useful long-term to help students recognize that multiplying by 10 (or any positive power of 10) will shift each digit one or more place values to the left. With this foundation, students will more easily build understanding of multiplication and division with decimal numbers when encountered later.

Example 4

Common Pitfall:	**When thinking about the value of a fraction, some adults will tell students, "The larger the bottom number (denominator), the smaller the fraction."** Although this is true when working with unit fractions (those with a numerator of 1), this claim causes much confusion when the numerator is not 1. For instance, using this "rule of thumb," students will incorrectly reason that 3/5 is smaller than 1/3 because 5 is larger than 3.
Best Practice:	More useful than rules that skirt understanding are questions that help students focus attention on making sense of unit fractions. Some to consider include: "What is the unit fraction I am working with?," "How many of these units do I have?," "Where does this fraction belong on a number line?," and "How might I create equivalent fractions with common units?" (the latter being important for comparing fractions with unlike denominators and building foundations for fraction operations). Relatedly, it is important when working with fractions to refrain from referring to the "top number" or "bottom number," because the entire fraction is a number.

Example 5

Common Pitfall:	**Many adults read decimal numbers without reference to place value.** For example, 3.2 is read as "three point two." In everyday life, this is efficient and generally not problematic. However, for students learning to make sense of decimal notation—who need to recognize decimals as fractions with denominators that are powers of 10—there must be more precision in how we talk about these numbers.
Best Practice:	Hearing "three and two-tenths" instantly communicates the meaning of the decimal value and its connection to fraction notation (making it easier to recognize 3.2 as equivalent to 3 2/10).

Some Common Misconceptions

Before we dive into general strategies for eliciting and making productive use of misconceptions, let's take time to examine some common misconceptions and ways teachers might engage students in working through these. And let me acknowledge up front, before I became a teacher of mathematics and learned to focus on reasoning and sense making, my own understanding of some of these topics was at best superficial, because most of my mathematical education was about answer-getting. Approach these with your own sense of curiosity as a lifelong learner, too!

Counting Teen Numbers

What is the misconception?

11

When looking at the number written as 11, it is natural for early learners to read it as "one one" or "ten one."

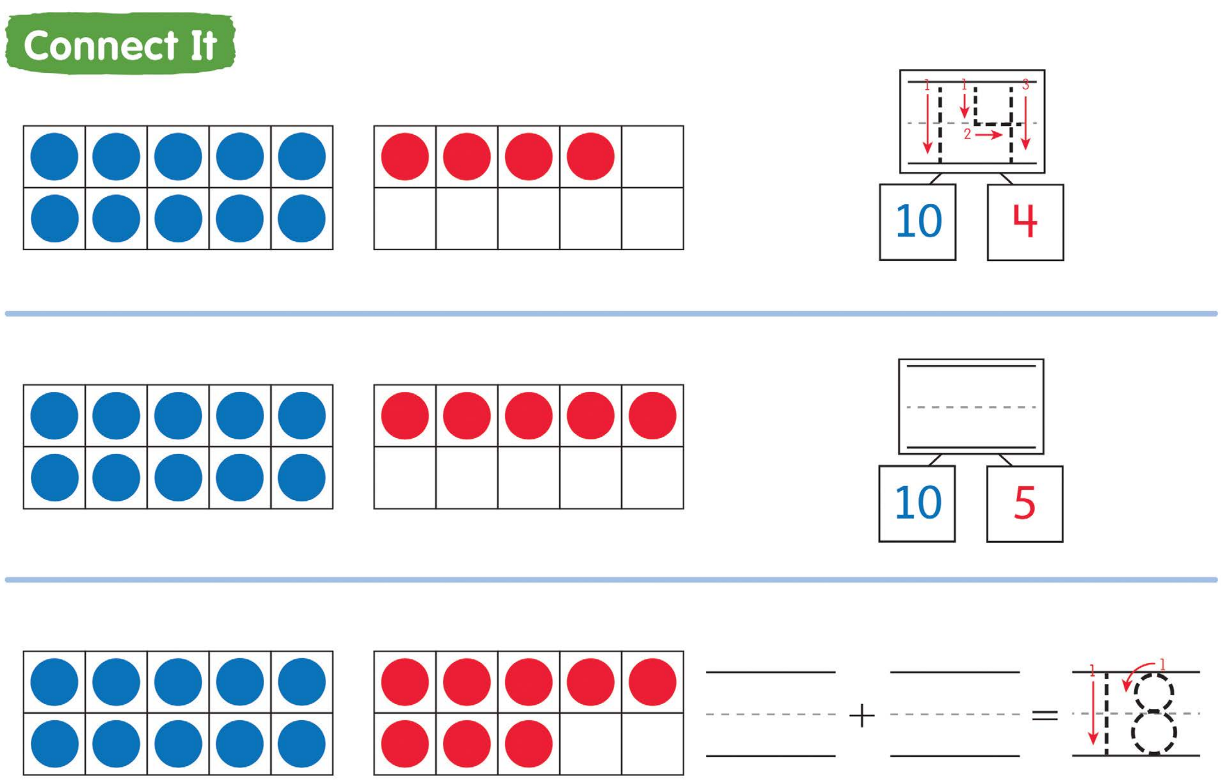


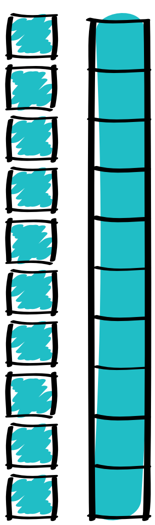
Figure 3. Connecting Ten Frames and Teen Numbers from *i-Ready Classroom Mathematics* (Grade K)

How/why do students develop this misconception?

The first of these, "one one," is rooted in a reasonable misconception related to emerging understanding of how the place value system works.

The second of these, "ten one," has more to do with a peculiar convention in English for naming values between 10 and 20 than with an actual misconception. It is conceptually correct to think "ten one," and the use of proper vocabulary ("eleven") will naturally happen through repetition.

How do you address this misconception?



To address the misconception of reading 11 as "one one," it is useful to provide opportunities for learners to use tools such as ten frames and objects that can easily be made into sets of ten (such as stacking cubes) to create sets of ten and sets with teen number values (see Figure 3 for an example from *Ready Classroom Mathematics*).

As they do so, focus more on their correct identification of the "ten" and "ones" rather than their correct use of English (or Spanish) number words. For example, it is fine to read 14 as "ten four" initially, as this reflects an understanding of tens and ones as distinct units, something that will build into formal place value understanding in Grade 1.

The facility with number words will come along with repeated use. Interestingly, students whose first language originates in Asia or the Middle East (e.g., Chinese, Korean, and Arabic, to name a few) have a built-in advantage when reading teen numbers, since the number words in these languages parallel the base-ten place-value system (which means that reading 11 as "ten one" is correct in these languages). Unfortunately, English, Spanish, and many other Latin-based languages require students to learn several new words in order to count to 20!

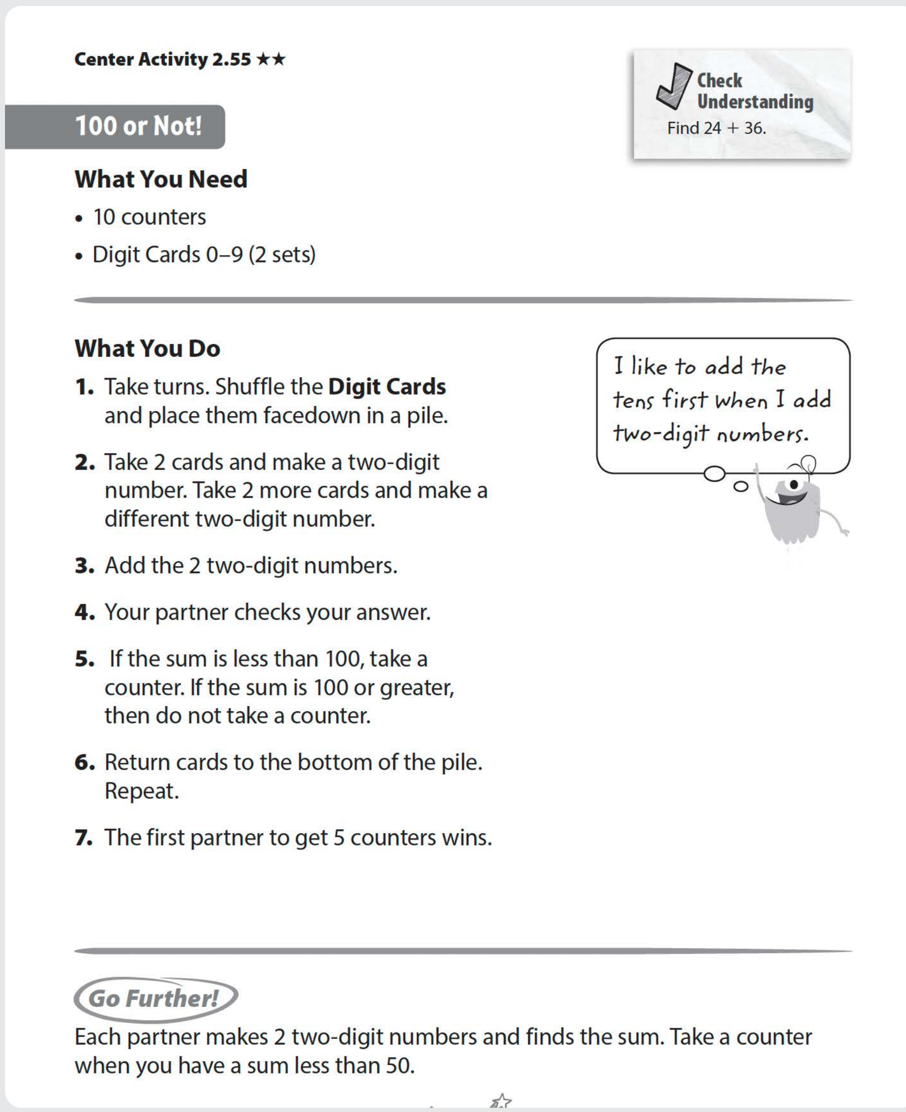
Meaning of Place Value Leading to Two-Digit Addition

What is the misconception?

$$\begin{array}{r} 34 \\ +\ 18 \\ \hline 412 \end{array}$$

When students begin to add beyond single-digit numbers, there will be misconceptions that arise related to how to name the result of their operations due to a still-fragile understanding of renaming units within the base-ten system. This is very common!

For example, when asked to find 34 + 18, a student might give an answer of 412 (can you guess why?). This is entirely reasonable and not cause for concern! Rather, it represents the opportunity for a discussion about the student's justification.



Center Activity 2.55 ★★

100 or Not!

Check Understanding
Find 24 + 36.

What You Need

- 10 counters
- Digit Cards 0–9 (2 sets)

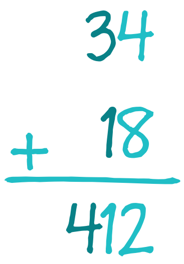
What You Do

1. Take turns. Shuffle the **Digit Cards** and place them facedown in a pile.
2. Take 2 cards and make a two-digit number. Take 2 more cards and make a different two-digit number.
3. Add the 2 two-digit numbers.
4. Your partner checks your answer.
5. If the sum is less than 100, take a counter. If the sum is 100 or greater, then do not take a counter.
6. Return cards to the bottom of the pile. Repeat.
7. The first partner to get 5 counters wins.

Go Further!
Each partner makes 2 two-digit numbers and finds the sum. Take a counter when you have a sum less than 50.

Figure 4. Prompt to Make a Sum Less Than 100 from *i-Ready Classroom Mathematics* (Grade 2)

How/why do students develop this misconception?



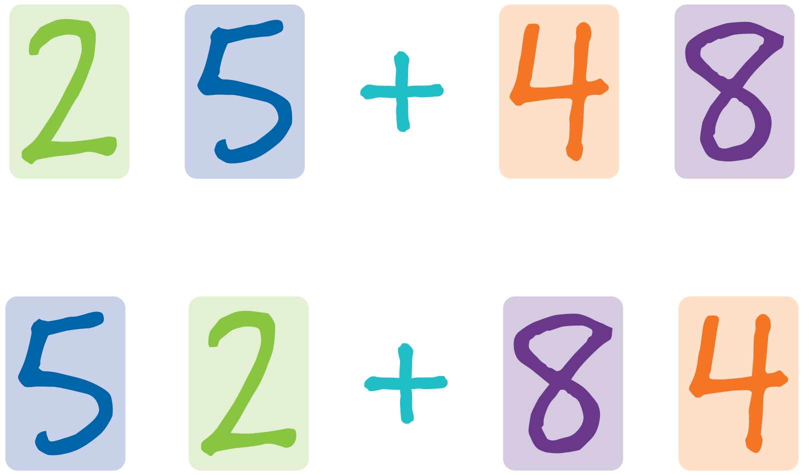
Students who are comfortable with single-digit addition but not yet proficient with place-value concepts will see an expression like 34 + 18 as two pairs of addition problems that for some reason are placed side by side. They will add 4 + 8 to get 12 (and write that down) and 3 + 1 to get 4 and place that next to the other sum to make 412. This demonstrates an understanding of single-digit addition without yet understanding the significance of place value.

How do you address this misconception?

A couple of prompts that can assist with moving students to make sense of regrouping and renaming values with base ten are:

- Asking students to show a given value in more than one way using base-ten blocks as well as numbers (see Figure 4). This builds flexibility with using tens and ones (and can be extended to larger place values).

- Challenging students to use *n* digits to create the largest or smallest value (e.g., "Using 2, 4, 5, and 8, create the largest sum: _____+ _____"). While some students may begin this task by guessing and checking, encourage them to notice features of the solutions that generate the largest or smallest value—what is it about the placement of digits that matters? What makes this powerful is asking students to explain and justify their reasons for placing each digit. After several "rounds" of the task, students will have had multiple opportunities to use and hear concepts of place value and arithmetic. This can later be extended to work with subtraction and multiplication.



Meaning and Use of Zero

What is the misconception?

The concept of zero is an abstraction invented by humans to allow for more efficient work with large quantities. Think about the difference between addition with Roman numerals versus with our familiar Hindu-Arabic system based on groups of (powers of) 10.

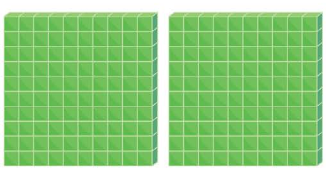
The critical difference between these two systems is the use of place value; whereas with Roman numerals each symbol has exactly one value (e.g., V always represents five), the Hindu-Arabic number system uses a set of only 10 symbols (0 through 9) that can take on a range of values depending on the place value in which they appear.

For instance, in 50, the "5" has a value of 5 x 10, whereas in 5,000, the "5" has a value of 5 x 1,000. It is the use of zero that makes it possible to differentiate one place value from another, serving as a placeholder when there is no quantity of a given place value.

MODEL IT

Fill in the blanks below.

3 Look at the blocks below.



The blocks show .... **hundreds**.

4 Think about ways to show 200 using **place value**.

0 hundreds + 0 tens + ... ones

0 hundreds + ... tens + 0 ones

.... hundreds + 0 tens + 0 ones

DISCUSS IT

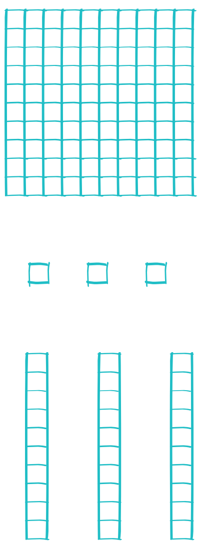
- How did you and your partner decide how many ones, tens, and hundreds would complete the statements?
- I think 20 tens and 2 hundreds are different because . . .
- I think 20 tens and 2 hundreds are similar because . . .

Figure 5. Exploration of Place Value and Zeros from *i-Ready Classroom Mathematics* (Grade 2)

How/why do students develop this misconception?

Because this is not intuitive, for an early learner it is not obvious why "0" is needed at all; if there are no objects to count, then there is no need for a number to represent them! Student misconceptions and misuse of zero become particularly evident when they are learning to work with numbers in the hundreds.

How do you address this misconception?



One way to reinforce the correct use of zeros is to ask students to generate several ways to represent a number in the hundreds, starting with exact hundreds (see Figure 5). The responses to such a prompt will give students opportunities to work through their misconceptions about zero with peer sharing and whole class discussions.

In addition, it is helpful to ask students to use a place-value chart (see Figure 6) to write and read numbers in more than one way. With the same example as in Figure 5, students can write "200" in the place-value chart and be asked to read it as many ways as possible.

It might help initially to allow students to also represent these different readings with base-ten blocks (e.g., regrouping the 2 hundreds as 20 tens), linking this to the activity above. The rigor of this activity can be increased gradually by introducing values like 230 (with non-zero digits in the hundreds place and tens place) and finally 245 (with non-zero digits in each place value).

Have students point to the hundreds place and say:

2 hundreds, 0 tens, 0 ones.

Hundreds	Tens	Ones
2	0	0

Have students point to the tens place and say:

20 tens, 0 ones.

Hundreds	Tens	Ones
0	20	0

Have students point to the ones place and say:

200 ones.

Hundreds	Tens	Ones
0	0	200

Figure 6. Place-Value Charts from *i-Ready Classroom Mathematics* (Grade 2)

Interpreting and Comparing Decimal Numbers

What is the misconception?

0.3

0.03

For many students, decimal numbers are initially a source of conceptual confusion because the digits look the same as whole numbers but do not "act" the same as whole numbers. One common misconception students have in their early work with decimals is that more digits means a greater value and fewer digits means a smaller value.

How/why do students develop this misconception?

3

30

This makes perfect sense because that pattern was learned from students' prior experiences with whole numbers; any two-digit whole number (10 to 99) is always greater than any single-digit whole number (0 to 9). It is understandable that when asked to compare 0.5 and 0.17, students might initially believe 0.17 is greater than 0.5 because there are two digits (17) compared to only one digit (5).

How do you address this misconception?

For students to revise their thinking, it is important to give them opportunities to make sense of decimal numbers for themselves. This is far more effective and long-lasting than simply stepping in to tell them what is correct and incorrect.

One effective strategy is to ask students to create visual representations for a set of decimal numbers using a hundredths grid (similar to a hundreds grid but with the entire grid having a value of 1 and each small square equivalent to 1/100 or 0.01). See Figure 7 for an example from *Ready Classroom Mathematics* (Grade 5). Some questions to ask students include:

- What happens to the digit "3" each time it is moved one place value to the right? Why?
- How are 0.3 and 0.03 related mathematically? Write each one as a fraction to help with your explanation.
- Using the same grid in model C, change the shading to show 0.07. Is 0.07 greater than or less than 0.3? Why?
- How might you convince your friend who missed class today that 0.07 is smaller than 0.3, even though 7 is more than 3?

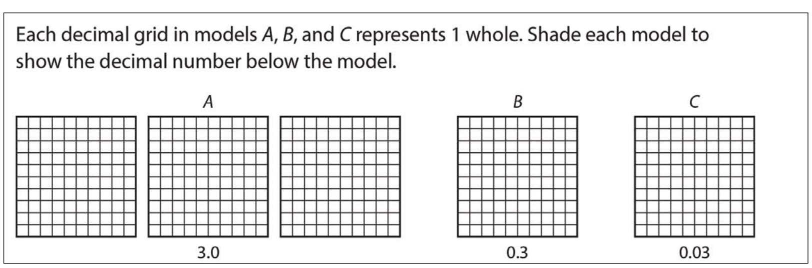


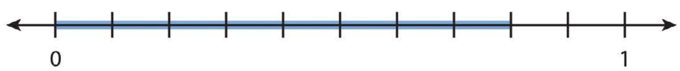
Figure 7. Representing Decimal Numbers with a Hundredths Grid from *i-Ready Classroom Mathematics* (Grade 5)

How do you address this misconception? (continued)

The number line is another representation to have students explore as they revise and refine their understanding of decimal numbers. The task in Figure 8 asks students to identify the misconception (referred to as an error) and explain how 0.08 can be correctly placed. This sort of error-analysis task is known to stimulate cognitive restructuring among learners as they first recognize and explain a common misconception (in this case, representing 8 hundredths as 8 tenths) and then must generate a representation that correctly shows 8 hundredths.

2 ANALYZE

Kiran showed 0.08 with the model below.



What is wrong with Kiran's model? Describe how the length she shaded relates to the length that she should have shaded. How can Kiran change her model to show 0.08?

Figure 8. Representing Decimal Numbers on a Number Line from *i-Ready Classroom Mathematics* (Grade 5)

Finally, the task in Figure 9 allows for student choice and creativity while assessing understanding of decimal place value. It is recommended to give students two to three minutes to try this on their own before asking them to share their thinking with a partner. While students are discussing and revising their ideas, the teacher can circulate around the room looking for (a) student competencies to call out in the moment and (b) student examples that might be productive to share with the class for discussion.

4 Starting with the number 23.5, place a 0 so that the new number is:

a) Equivalent to 23.5

b) Greater than 23.5

c) Less than 23.5

For each response, explain why your reasoning makes sense.

Figure 9. Have Students Place a 0 to Create Different Relationships to the Original Number

Dividing by a Fraction

What is the misconception?

Building on students' understanding of whole number division to include fractions, typically in Grade 6, requires anticipating misconceptions that have to do with both the concept of division and the concept of fractions. It is common when first asked to consider an expression such as 6 ÷ 1/2 that students misinterpret it as multiplication or as whole number division. A quick way to uncover such ideas is to show students the expression 6 ÷ 1/2 and ask them to:

- Give a situation that is represented by 6 ÷ 1/2
- Draw a model that represents 6 ÷ 1/2

How/why do students develop this misconception?

Most students (and even many adults whose mathematical learning emphasized procedures without conceptual understanding) will respond with scenarios and models that represent 6 x 1/2 or 6 ÷ 2. This reflects common misconceptions that (a) division makes things smaller, and (b) division is always about sharing (also referred to as a partitive model).

How do you address this misconception?

In order to challenge these misconceptions, teachers must intentionally provide opportunities for students to expand their understanding of division to include a measurement model (sometimes referred to as a quotative model) that builds on what they learned about unit fractions since Grade 3. The meaningful context of cutting wood boards to make game pieces gives students something concrete to aid their thinking and visualization. The use of a bar model (or area model) connects with students' earlier experiences using the same model for making sense of multiplication and division of whole numbers. As students work on this and similar problems, it is important to give them opportunities to articulate and clarify their thinking with small groups, partners, and the whole class. Some productive prompts to use include:

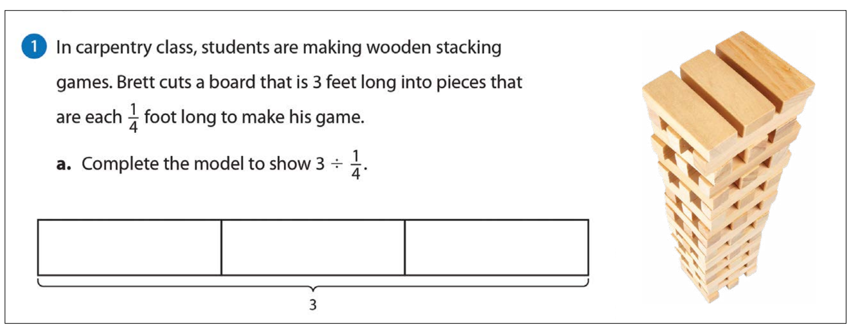


Figure 10. Making a Wooden Stacking Game from *i-Ready Classroom Mathematics* (Grade 6)

- What does the "3" in the model represent?
- How might we show the 1/4-foot-long pieces in the model?
- For each 1 foot of board, how many 1/4-foot-long pieces can be cut? How do you know this?

How do you address this misconception? (continued)

- When you complete the sentence, "3 ÷ 1/4 = ______" by finding the quotient, explain what the value represents. Why does this make sense in the context of the problem?
- Starting with the same 3-foot-long board, Brett's friend Lizette decides to cut her pieces into 3/4 foot each. Will she have more pieces or fewer pieces than Brett? Why?

Another strategy to help students clarify misconceptions about division by a fraction, particularly that division always makes things smaller, is to have students examine number strings to notice and name patterns. This sort of decontextualized exploration builds on the contextual problems like the one above and focuses students' attention on properties of number and arithmetic.

For each equation involving division by a fraction, draw a visual model to represent it. Then look for and describe any patterns that you find.

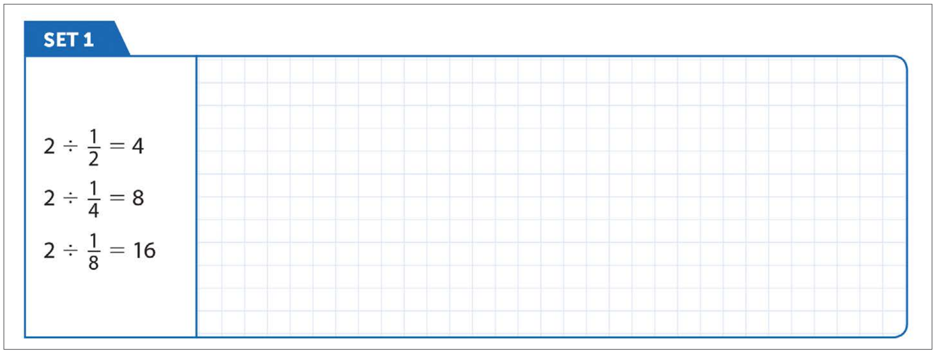


Figure 11. Examining Patterns in Division by a Fraction from *i-Ready Classroom Mathematics* (Grade 6)

Engaging in repeated reasoning about a specific mathematical operation—in this case, division by a fraction—gives students the opportunity to notice new relationships and think more deeply about their understanding. Depending on what has preceded this task, it might be helpful to include whole number divisors, too (e.g., 2 ÷ 2 and 2 ÷ 1).

As students share their thinking, communicate a genuine sense of curiosity about their insights and press them for clarification when their statements lack precision (or have their peers do so).

Once the idea has been shared that division by a fraction (a) seems to make things larger and (b) is related to the product of the dividend and the denominator of the division, be sure to ask students how this can be justified. In other words, why are such statements true (and are they always true)? Early in the process of sense making, it is not necessary to obtain complete answers to the request for justification, but this should always be the long-term goal.

Similar to what is done with early versions of a writing assignment in English or history classes, students' rough draft thinking in mathematics offers a place to begin exploring, sense making, and revising their understanding.

Such an approach communicates to students that learning mathematics is a generative process of developing, refining, and extending understanding rather than a static state of knowing or not knowing.

Productive Strategies for Eliciting and Addressing Misconceptions

The key to eliciting and making the best use of student misconceptions is to frame students' initial ideas about a concept or relationships as rough draft thinking (Jansen, 2020; Jansen, Cooper, Vascellaro, & Wandless, 2016).

Similar to what is done with early versions of a writing assignment in English or history classes, students' rough draft thinking in mathematics offers a place to begin exploring, sense making, and revising their understanding.

Such an approach communicates to students that learning mathematics is a generative process of developing, refining, and extending understanding rather than a static state of knowing or not knowing. As Barnes (2008) expressed:

"The construction of knowledge is essentially a social process. How teachers behave in lessons, and particularly how they receive and use their pupils' written and spoken contributions, is crucial in shaping how pupils will set about learning and therefore what they will learn. It is by the way that a teacher responds to what a pupil offers that he or she validates—or indeed fails to validate—that pupil's attempts to join in the thinking."

In this sense, promoting students' rough draft thinking in mathematics as a means to elicit and address misconceptions positions students with authority over the content (i.e., the power to author their understanding).

Students come to recognize and value their own ways of thinking and reasoning with and about mathematical ideas and take ownership of this knowledge. This is an important element of culturally responsive mathematics teaching (CRMT), which aims to value and build on students' sense of identity in relation to their learning of mathematics (see Ellis, 2018 for more about CRMT).

For students to be willing to engage in rough draft thinking, they must perceive the classroom to be a safe space in which to offer ideas that may be only partially formed and about which they may be unsure of the correctness.

Steuer, Rosentritt-Brunn, and Dresel (2013) found that when teachers created what they refer to as a positive climate for errors, students not only shared their thinking more freely but also were more willing to grapple with ideas of other students and revise misconceptions after receiving feedback from others. As the researchers explain:

"The degree to which students initiate cognitive processes and behaviors aimed to specifically overcome possible misconceptions underlying errors depended on a collective analysis of errors in the classroom context and teachers' specific support following student errors" (Steuer, Rosentritt-Brunn, & Dresel, 2013, p. 207).

University of Delaware mathematics educator Amanda Jansen and the teachers with whom she collaborates suggest three principles that support rough draft thinking in mathematics along with classroom practices that encourage this thinking (see Table 1 on page 22).

Note how the principles and practices resonate with the four "Rights of the Learner" (see Figure 12) by positioning students as mathematical thinkers and sense makers. It is an issue of equity that for too many students, mathematics feels like a zero-sum game that they repeatedly lose. When the correctness of answers is valued over the quality of students' thinking, we set narrow boundaries on what counts as being "smart" in mathematics.

Rights of the Learner

Elementary teacher of mathematics Olga Torres has developed "Four Rights of the Learner (RotL) in the classroom: (1) the right to be confused; (2) the right to claim a mistake; (3) the right to speak, listen, and be heard; and (4) the right to write, do, and represent only what makes sense" (Kalinec-Craig, 2017).

This framework is a response to traditional 20th-century practices of teaching mathematics that positioned learning as memorization and mimicry of procedures, which denied students opportunities to make sense of problems and develop strategies based on their own thinking.

The dehumanizing and inequitable mathematics learning experiences and outcomes that result from these 20th-century practices are well known and widespread, including many students thinking that they were not "good" in math because they didn't understand the one strategy they were shown—or worse, their correct strategies were marked wrong because they were different from what the text or the teacher expected (Ellis & Berry, 2005; Ellis, 2008; Gutiérrez, 2018).

By articulating and respecting the RotL in the classroom, Ms. Torres frames mathematics as a subject that has coherence, makes sense, and positions her students as capable learners who can use their ways of thinking, sense making, and communicating to build and extend their mathematical understandings.

The teacher's role is to learn from and with his or her students as they explore, struggle, and revise their mathematical ideas within their classroom community.

The book Reimagining the Mathematics Classroom: Creating and Sustaining Productive Learning Environments *by Yeh, Ellis, and Hurtado (2017) offers examples of classrooms in which these principles guide powerful mathematics teaching and learning.*

Figure 12. Rights of the Learner

Principles and Practices That Support Rough Draft Thinking in Mathematics

(adapted from Jansen et al., 2016)

① Foster a Culture Supportive of Intellectual Risk Taking

Thank you for getting us started trying to make sense of this. *Great rough draft thinking.*	Call out students' initial ideas as "rough drafts."
Can you explain that to me again? I didn't quite understand. *Please clarify why . . .*	Have students share their thinking with one another and ask each other for justifications and clarifications.
Can someone build on _____'s thinking? *I wonder what happens if . . .*	Model and encourage a nonevaluative manner of responding to others' ideas.

② Promote the Belief That Learning Mathematics Involves Revising Understanding

Marco, I appreciate how you revised your initial idea.	Celebrate when students revise their thinking.
On your whiteboard, show your partner how you arrived at the answer.	Promote peer feedback through the use of whiteboards, different colors of pencil/ink, and whole class discourse.
I don't understand what you did here [point to a specific part of their work]. Can you explain it again in another way? *Why exactly did you use this symbol?*	Press students to be more precise with their representations, explanations, and use of symbols.

③ Raise Students' Statuses by Expanding What Counts as a Valuable Contribution

I really like how you listened to each other and asked questions to clarify your thinking. *Oh, you noticed an interesting pattern. I wonder how that connects to_____.* *Nice job trying different strategies until you could make sense of this.*	Assign competence to student contributions other than "correctness" or "quickness" (e.g., use of representations, posing questions, exhibiting productive struggle, noticing a pattern or connection, listening actively to a peer, etc.).

Table 1. Principles and Practices That Support Rough Draft Thinking in Mathematics

1 Foster a Culture Supportive of Intellectual Risk Taking

The first principle, "Foster a Culture Supportive of Intellectual Risk Taking" (Jansen et al., 2016), is about communicating to students that their thinking is valued within the context of learning mathematics.

In order to promote this stance, teachers and students should engage in activities that model ongoing curiosity and inquisitiveness about mathematics (e.g., "Why does that work?," "Is that true all the time?," "I wonder what happens if ____."). This includes politely critiquing the reasoning of others and using multiple means of representing and justifying their developing understandings. Even when an incomplete understanding is shared, it can provide an opportunity for all students to further their own thinking.

For example, "Here is how one student [use their name if possible] started this problem: [show student's work or have them share it]. What do you think they were thinking? And what might we do next?"

Such discussions position the student's initial ideas as valued and useful for the learning of the entire class. This approach requires not only structures that support student–student discourse (such as the Try–Discuss–Connect routine) but also that the teacher's response to student ideas is initially nonevaluative.

Once the teacher signals an evaluation of an idea (e.g., "Yes," "That's right," "Not quite," "Hmm, I don't know"), students with different ideas will be more reluctant to share, and the student who offered the idea will stop listening to other ideas.

2 Promote the Belief That Learning Mathematics Involves Revising Understanding

The second principle, "Promote the Belief That Learning Mathematics Involves Revising Understanding" (Jansen et al., 2016), involves practices that communicate to students that learning mathematics is always a work in progress. Whether during a lesson, a unit, a school year, or a lifetime, those who understand mathematics best are always ready and willing to revise their understandings when presented with new information and insights.

Revising understanding can be done by explicitly and intentionally pointing out when someone (including the teacher) recognizes a new insight or clarifies an earlier belief. Ask questions like those below to communicate a sense of wonder and excitement around updating one's thinking.

Who revised their thinking about ____?

Did anyone else just notice ____?

Using statements such as the ones below can convey an interest in processes of revision and sense making.

I want to know how your thinking has changed. Tell me how "I used to think ____, but now I know _____."

By explicitly teaching students to provide meaningful feedback to peers, orally and in writing, teachers further communicate that learning mathematics is always a work in progress.

Many teachers tend to "step in" and offer a helpful correction or suggestion when they see a student struggling to convey an idea or revise a misconception. While well-intentioned, this has the effect of shutting down student cognition and discouraging their perseverance; students learn to just wait for the teacher to tell them what's right.

In writing, sentences such as those below offer students ways to give feedback that prompts revision and clarification.

- What if you ____?
- Can you explain it or show me in another way?
- I don't understand what you said/did when you ____.

Some teachers have students working together using different-colored pencils or pens so it is clear how they have supported one another's thinking. All these practices help normalize the belief that learning mathematics involves revising understanding over time.

Teachers must exercise caution in how the process of revising thinking is supported. Many teachers tend to "step in" and offer a helpful correction or suggestion when they see a student struggling to convey an idea or revise a misconception. While well-intentioned, this has the effect of shutting down student cognition and discouraging their perseverance; students learn to just wait for the teacher to tell them what's right. It's more effective to provide feedback in the form of questions or prompts to focus students' attention on a specific feature of their work.

University of Connecticut mathematics educator Megan Staples collaborated with middle school mathematics teachers for two years to develop strategies that would support students with the justification and revision of their mathematical thinking. In reflecting on what was learned, the teachers had this to say (Cioe, King, Ostien, Pansa, & Staples, 2015):

"Our work was most effective when we built on and developed students' thinking, even when it did not match how we were thinking or what we thought was the 'best' approach. If we did not consistently work to build on students' ideas, it would undermine efforts to get students to generate and then develop their own ideas toward a more complete justification. This work required careful listening by the teacher and making deliberate efforts to develop students' ideas. The main point is that justification is about reasoning: It cannot be the teacher's reasoning; it has to be the students' reasoning."

That last sentence sums it up nicely: "It has to be the students' reasoning."

3 Raise Students' Statuses by Expanding What Counts as a Valuable Contribution

The third principle, "Raise Students' Statuses by Expanding What Counts as a Valuable Contribution" (Jansen et al., 2016), is an important method to reduce status issues in the mathematics classroom that result from traditional practices of overvaluing speed and correctness as indicators of "smartness."

Having the teacher and other students regularly attend to and acknowledge a wide range of competencies as valuable to the class's learning of mathematics is an effective way to broaden notions of smartness and create greater inclusion.

One way to begin redefining mathematical competence is to ask students to brainstorm all the ways someone can contribute to their own learning and the learning of their peers. Notice the focus is on learning, not on answer-getting. It might help to share a couple of examples, like being a good listener, asking clarifying questions when working with a partner, thinking creatively about how to approach a problem, and being able to picture or represent an idea in ways that help others understand it.

Post the list of competencies so students have access to it (and even print it out for each student to have on hand), and provide examples of sentence stems that both the teacher and students can use to call out these new indicators of "smartness" during every lesson, such as:

- ____ asked questions that helped us figure out ____.
- I like how ____ represented the problem by ____.
- When ____ pointed out the pattern that ____, I finally made sense of why ____.
- I appreciate that ____ shared thinking that was not complete because it gave me an idea that helped make sense of ____.
- ____ really listened to what I was trying to say and asked me good questions to help me organize my thoughts better.

What's important is that the statements are specific and authentic. Telling someone simply "Good job" or "You're smart" does not help them recognize what action is being valued and why.

For more information about expanding notions of competency in math, see this website curated by educators at High Tech High: mathagency.org/status-mindset-resources

Principles and Practices That Support Rough Draft Thinking in Mathematics *(adapted from Jansen et al., 2016)*

Foster a Culture Supportive of Intellectual Risk Taking

What Do You Do?

- Call out students' initial ideas as "rough drafts."
- Have students share their thinking with one another and ask each other for justifications and clarifications.
- Model and encourage a nonevaluative manner of responding to others' ideas.

What Do You Say/Ask?

- *Why does that work?*
- *Is that true all the time?*
- *Did anyone else notice ____?*
- *I wonder what happens if ____?*

Promote the Belief That Learning Mathematics Involves Revising Understanding

What Do You Do?

- Celebrate when students revise their thinking.
- Publicly recognize and celebrate the revision of a student's thinking when it occurs.
- Promote peer feedback through the use of whiteboards, different-colored pencils/ink, and whole class discourse.
- Press students to be more precise with their representations, explanations, and use of symbols.

What Do You Say/Ask?

- *What if you ____?*
- *Who revised their thinking about ____?*
- *Can you explain it or show me in another way?*

Raise Students' Statuses by Expanding What Counts as a Valuable Contribution

What Do You Do?

- Ask students to brainstorm all the ways someone can contribute to their own learning and the learning of their peers.
- Assign competence to student contributions other than "correctness" or "quickness," e.g.:
  - Use of representations
  - Posing questions
  - Noticing a pattern or connection
  - Exhibiting productive struggle
  - Listening actively to a peer
  - Sharing "in-progress" thinking

What Do You Say/Ask?

- *____ asked questions that helped us figure out ____.*
- *I like how ____ represented the problem by ____.*

How else could we represent this situation?

Can you politely critique the reasoning of [student's name]?

Can you explain your thinking and justify your reasoning?

Cautions

Avoid evaluating student ideas. Once the teacher signals an evaluation of an idea ("Yes," "That's right," "Not quite," "Hmm, I don't know"), students with different ideas will be more reluctant to share, and the student who offered the idea will stop listening to other ideas.

I want to know how your thinking has changed.

I don't understand what you said/did when you ____.

Complete this sentence: I used to think ____, but now I know ____.

What do you think [student's name] was thinking? And what might we do next?"

Cautions

Avoid "stepping in" and offering a helpful correction when you see a student struggling to convey an idea or revise a misconception. While well-intentioned, this has the effect of shutting down student cognition and discouraging their perseverance; students learn to just wait for the teacher to tell them what's right. It's more effective to provide feedback in the form of questions or prompts to focus the student's attention on a specific feature of their work.

____ really listened to what I was trying to say and asked me good questions to help me organize my thoughts better.

When ____ pointed out the pattern that ____, I finally made sense of why ____.

I appreciate that ____ shared thinking that was not complete because it gave me an idea that helped make sense of ____.

Cautions

Make sure your statements are specific and authentic. Telling someone simply "Good job" or "You're smart" does not help them recognize what action is being valued and why. Instead, use statements like those at the left that specifically state what they did to contribute to their mathematical understanding and that of their peers.

Concluding Thoughts

Using misconceptions to develop deeper understandings in mathematics can be challenging when it is new to teachers and/or students who have become accustomed to mathematics as mimicking procedures without understanding.

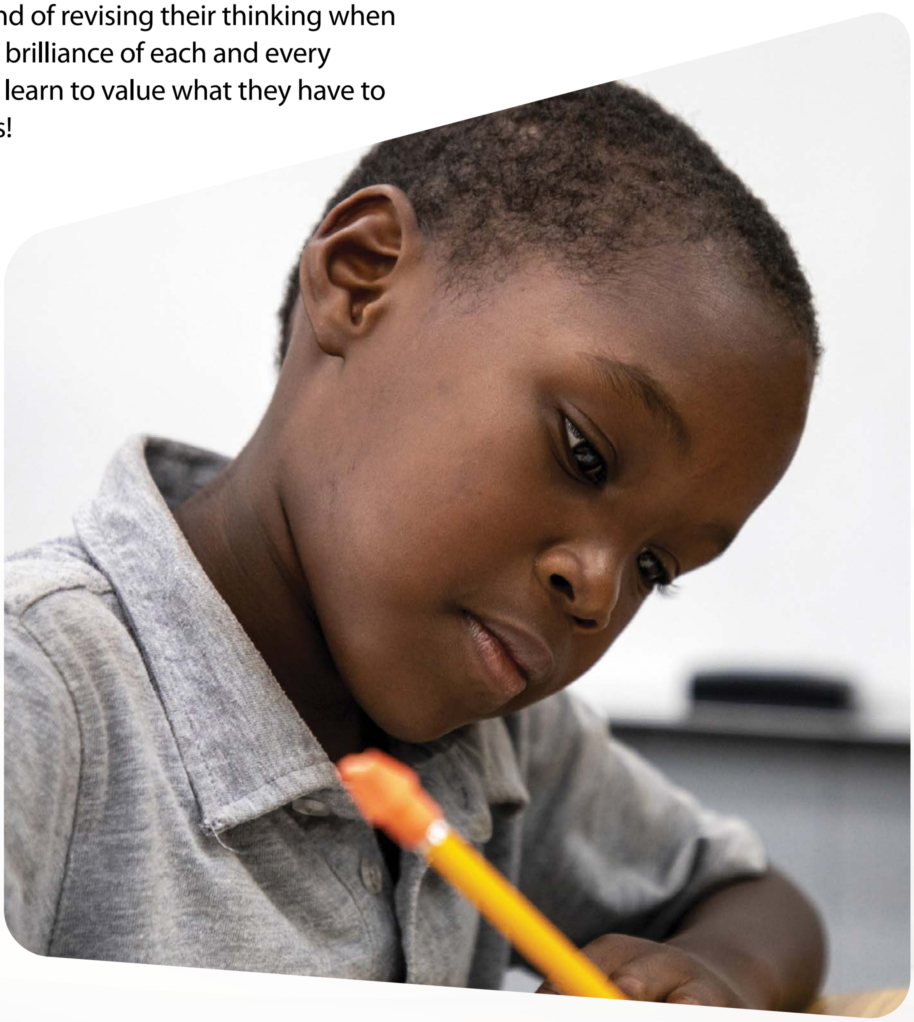
However, in the 21st century—an age when computers can perform millions of calculations per second and are being taught to solve routine mathematical problems that once took humans hours, days, and weeks—students need to become flexible with their understanding of mathematics so they can engage in the creative, non-routine reasoning that computers are unable to do. This means it is imperative that students learn to make sense of key concepts and relationships through processes of reasoning, justification, and revision.

Eliciting and making use of misconceptions by perceiving students' thinking as a "rough draft" that is always in the process of revision is a powerful way to support deep learning in mathematics (Jansen, 2020).

The more teachers use such strategies to engage students in exploring multiple representations, conceptual connections, and coherence, the more their own knowledge of mathematics will grow.

With these strategies, teachers and students both learn the importance and the power of questioning what they know and of revising their thinking when faced with new insights. It can also show the brilliance of each and every student as a mathematical thinker. When we learn to value what they have to share, our students have so much to teach us!

Eliciting and making use of misconceptions by perceiving students' thinking as a "rough draft" that is always in the process of revision is a powerful way to support deep learning in mathematics.



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